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Estimates of low Sobolev norms of solutions for some nonlinear evolution equations

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ABSTRACT

In this work we obtain results on the estimates of low Sobolev norms for solutions of some nonlinear evolution equations, in particular we apply our method for the complex modified Korteweg–de Vries type equation and Benjamin–Ono equation.

1. Introduction

In this paper we describe some results on estimates of low $\dot{H}^s$-norm of solutions u for models such that

$$\left\|u(t)\right\|_{\dot{H}^k} \leqslant \left\|u(0)\right\|_{\dot{H}^k} + c\left\|u(0)\right\|_{L^2}^n, \qquad \left\|u(t)\right\|_{L^2} \leqslant \left\|u(0)\right\|_{L^2}, \quad \forall t \in \mathbb{R}, \tag{1.1}$$

with scaling

$$\tilde{u}(x,t) = \lambda^{\beta_1} u\left(\lambda^{\beta_2} x, \vartheta t\right), \tag{1.2}$$

where $k > 0$, $n > 1$, $\beta_2 > 0$, $\vartheta \in \mathbb{R}$, $\vartheta \neq 0$ and

$$\beta_1 = \left(\frac{1}{2} + \frac{k}{n-1}\right)\beta_2. \tag{1.3}$$

The scaling solution satisfies $\|\tilde{u}(t/\vartheta)\|_{\dot{H}^\theta} = \lambda^{r(\theta)}\|u(t)\|_{\dot{H}^\theta}$, where $r(\theta) = \beta_1 + \beta_2\theta - \beta_2/2$ and if $u = \tilde{u}$ in (1.1) then $\|u(t)\|_{\dot{H}^k} \leqslant \|u_0\|_{\dot{H}^k} + c\lambda^{nr(0)-r(k)}\|u(0)\|_{L^2}^n$, therefore we need that $nr(0) - r(k) = 0$, and this equality gives the condition (1.3).

An example of this type of model is the complex modified Korteweg–de Vries type equation (complex mKdV)

$$\begin{cases} \partial_t u + b\partial_x^3 u + d|u|^2\partial_x u + eu^2\partial_x \bar{u} = 0, \\ u(x,0) = u_0, \end{cases} \tag{1.4}$$

where u is a complex valued function and b, d and e are real parameters with $b \cdot e \neq 0$.

This model was proposed by Hasegawa and Kodama in [2,3] to describe the nonlinear propagation of pulses in optical fibers.

The flow associated to the IVP (1.4) leaves the following quantity

$$I_1(u) = \int_{\mathbb{R}} \left|u(x,t)\right|^2 dx, \tag{1.5}$$

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conserved in time. Also, when $b \cdot e \neq 0$ we have the following time invariant quantity

$$\int_{\mathbb{R}} \left|\partial_x u(x,t)\right|^2 dx + k_0 \int_{\mathbb{R}} \left|u(x,t)\right|^4 dx = \int_{\mathbb{R}} \left|\partial_x u(x,0)\right|^2 dx + k_0 \int_{\mathbb{R}} \left|u(x,0)\right|^4 dx, \tag{1.6}$$

where $k_0 = -(e+d)/(6b)$. The complex mKdV equation satisfies the conditions (1.1)–(1.3) with $\beta_1 = \beta_2 = 1$, $k = 1$, $n = 3$ (see Proposition 3.2).

Other example that also we consider is the Benjamin–Ono (BO) equation

$$\begin{cases} \partial_t u + H u_{xx} = k_0 u u_x, & x, t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.7}$$

where H denotes the Hilbert transform on $\mathbb{R}$ defined by $\widehat{(Hf)}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi)$. The BO equation (T.B. Benjamin [1], and H. Ono [4]) arises in the study of unidirectional propagation of nonlinear dispersive waves. We have the following conservations laws for solutions of the BO equation (1.7)

$$\int_{\mathbb{R}} \left|u(x,t)\right|^2 dx = \int_{\mathbb{R}} \left|u(x,0)\right|^2 dx, \tag{1.8}$$

and

$$\int_{\mathbb{R}} \left|D_x^{1/2} u(x,t)\right|^2 dx + \frac{k_0}{3} \int_{\mathbb{R}} u(x,t)^3\, dx = \int_{\mathbb{R}} \left|D_x^{1/2} u(x,0)\right|^2 dx + \frac{k_0}{3} \int_{\mathbb{R}} u(x,0)^3\, dx. \tag{1.9}$$

The BO equation satisfies the conditions (1.1)–(1.3) with $\beta_1 = \beta_2 = 1$, $k = 1/2$, $n = 2$ (see Proposition 3.5).

We obtain a new estimate for the $\dot{H}^s(\mathbb{R})$ norm, $0 \leqslant s \leqslant 1$, of the solutions of (1.4). We also obtain a new estimate for the $\dot{H}^s(\mathbb{R})$ norm, $0 \leqslant s \leqslant 1/2$, of the solutions of (1.7).

The main results in this work are the following:

Theorem 1.1. *Let $u(t) \in H^{k+1}(\mathbb{R})$ be solution of some differential equation satisfying the condition* (1.1) *and with scaling functions* (1.1)–(1.3), *then for all $\theta \in [0,k]$ is*

$$\left\|u(t)\right\|_{\dot{H}^\theta} \leqslant n\left\|u(0)\right\|_{\dot{H}^\theta} + c\psi^{\theta/2+k/(2n-2)}\left\|u(0)\right\|_{L^2}^{\sigma}, \tag{1.10}$$

where

$$\sigma = 1 + \frac{n-1}{k}\theta, \quad \text{and} \quad \psi = 1 + \frac{e^{G_{u(0)}(k)}}{\|u(0)\|_{L^2}^{2(n-1)/k}},$$

and $G_{u(0)}(k)$ as defined in (1.20).

Now we consider the following IVP for all $k_0 \in \mathbb{R}$, $k_0 \neq 0$

$$\begin{cases} \partial_t u + Lu + k_0 R(u) = 0, \\ u(x,0) = u_0, \end{cases} \tag{1.11}$$

where Lu is the linear part and $R(u)$ is the nonlinear part of (1.11) and we suppose that $u(t)$ satisfies

$$\left\|u(t)\right\|_{\dot{H}^k} \leqslant \|u_0\|_{\dot{H}^k} + c|k_0|^{\Psi}\|u_0\|_{L^2}^n, \qquad \left\|u(t)\right\|_{L^2} \leqslant \|u_0\|_{L^2}, \tag{1.12}$$

where $\Psi > 0$. Note that, if we have (1.11) and (1.12) with $k_0 = 1$ and if moreover

$$R(\lambda u) = \lambda^{1+(n-1)/\Psi} R(u), \tag{1.13}$$

then we have (1.11) and (1.12) with $k_0 \neq 1$. For the complex mKdV and Benjamin–Ono equations is $\Psi = 1$ (see Propositions 3.2 and 3.5).

Theorem 1.2. *Let $u(t) \in H^{k+1}(\mathbb{R})$ be solution of IVP* (1.11) *with scaling functions* (1.2), (1.3) *and satisfying* (1.12) *and* (1.13), *then for all $\theta \in [0,k]$ is*

$$\left\|u(t)\right\|_{\dot{H}^\theta} \leqslant \|u_0\|_{\dot{H}^\theta} + c\psi^{\theta}\|u_0\|_{L^2}^{\sigma}, \tag{1.14}$$

where

$$\sigma = 1 + \frac{n-1}{k}\theta, \quad \text{and} \quad \psi = 1 + \frac{e^{G_{u(0)}(k)}}{\|u(0)\|_{L^2}^{2(n-1)/k}},$$

where $G_{u(0)}(k)$ was defined in (1.20).

In particular we have:

Theorem 1.3. *Let $u_0 \in H^2(\mathbb{R})$ and $u(t)$ be a solution of IVP (1.4) with initial data u_0, then for all $\theta \in [0,1]$, $t \in \mathbb{R}$*

$$\|u(t)\|_{\dot{H}^\theta} \leqslant \|u_0\|_{\dot{H}^\theta} + c\psi^\theta \|u_0\|_{L^2}^{1+2\theta}, \tag{1.15}$$

where $\psi = 1 + e^{G_{u_0}(1)}/\|u_0\|_{L^2}^4$.

And

Theorem 1.4. *Let $u_0 \in H^1(\mathbb{R})$ and let $u(t)$ be a solution of IVP (1.7) with initial data u_0, then for all $\theta \in [0,1/2]$, $t \in \mathbb{R}$*

$$\|u(t)\|_{\dot{H}^\theta} \leqslant \|u_0\|_{\dot{H}^\theta} + c\psi^\theta \|u_0\|_{L^2}^{1+2\theta}, \tag{1.16}$$

where $\psi = 1 + e^{G_{u_0}(1/2)}/\|u_0\|_{L^2}^4$.

Notation. The notation to be used is the standard in PDE. We will use the Lebesgue space–time $L_x^p\mathcal{L}_\tau^q$ endowed with the norm

$$\|f\|_{L_x^p\mathcal{L}_\tau^q} = \big\|\|f\|_{\mathcal{L}_\tau^q}\big\|_{L_x^p} = \left(\int_{\mathbb{R}}\left(\int_0^\tau |f(x,t)|^q\,dt\right)^{p/q} dx\right)^{1/p}.$$

We will use the notation $\|f\|_{L_x^pL_t^q}$ when the integration in the time variable is on the whole real line. The notation $\|u\|_{L^p}$ is used when there is no doubt about the variable of integration. We define the unitary group $V(t)u_0$ as the solution of the linear IVP

$$\begin{cases} \partial_t u + ia\partial_x^2 u + b\partial_x^3 u = 0, & x,t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.17}$$

in this way

$$\widehat{V(t)u_0}(\xi) = e^{it(b\xi^3 + a\xi^2)}\widehat{u_0}(\xi), \tag{1.18}$$

when $a = 0$ we will use the notation $V(t) := U(t)$.

We also will use the notation

$$\Lambda_{\theta,l}(u) = \int_{\mathbb{R}} \left(l + \xi^2\right)^\theta \log\left(l + \xi^2\right)\left|\widehat{u}(\xi)\right|^2 d\xi, \tag{1.19}$$

where $l \geqslant 0$ is a constant, and when $l \geqslant 1$, $\Lambda_{\theta,l}(u) = |||u|||_{\theta,l}^2$ and $\Lambda_{\theta,1}(u) = |||u|||_\theta^2$.

We define

$$G_u^l(\theta) = \frac{\Lambda_{\theta,l}(u)}{\|u\|_{\dot{H}^\theta}^2} \quad \text{and} \quad G_u(\theta) := G_u^0(\theta), \tag{1.20}$$

and when $u = u_0$ is an initial data, we will use

$$G(\theta) := G_{u_0}(\theta). \tag{1.21}$$

If $\tilde{u}(x) = \lambda_1 u(\lambda_2 x)$ then it is not hard to show that

$$G_{\tilde{u}}(\theta) = \log\lambda_2^2 + G_u(\theta). \tag{1.22}$$

2. Some inequalities in Sobolev spaces

To prove Theorems 1.1 and 1.2, we need the following results.

Lemma 2.1. *Let $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and let $f(\xi) > 0$ and $g(\xi) \geqslant 0$ be two functions such that for all $\theta \in [\alpha,\beta]$, for all $\xi \in \mathbb{R}$ holds,*

$$f(\xi)^\theta g(\xi) < h_1(\xi), \tag{2.23}$$

and for all $\theta \in (\alpha,\beta)$, for all $\xi \in \mathbb{R}$:

$$f(\xi)^\theta \left|\log\{f(\xi)\}\right|^j g(\xi) < h_2(\xi), \quad j = 1,2, \tag{2.24}$$

where $h_j \in L^1(\mathbb{R})$, $j = 1,2$, is independent of θ.

Let $F(\theta)=\int_{\mathbb{R}} f(\xi)^\theta g(\xi)\,d\xi$, then if $\theta_1\in(\alpha,\beta)$ and $\theta_2\in[\alpha,\theta_1]$ we have

$$F(\theta_1)\leqslant F(\theta_2)\exp\left\{\frac{(\theta_1-\theta_2)F'(\theta_1)}{F(\theta_1)}\right\}, \tag{2.25}$$

and

$$F(\theta_1)\geqslant F(\theta_2)\exp\left\{\frac{(\theta_1-\theta_2)F'(\theta_2)}{F(\theta_2)}\right\}, \tag{2.26}$$

where $F'(\theta)=\int_{\mathbb{R}} f(\xi)^\theta \log\{f(\xi)\}g(\xi)\,d\xi$.

Proof. By (2.24) we obtain

$$F'(\theta)=\int\limits_{\mathbb{R}} f(\xi)^\theta \log\bigl\{f(\xi)\bigr\}g(\xi)\,d\xi,\qquad F''(\theta)=\int\limits_{\mathbb{R}} f(\xi)^\theta \log^2\bigl\{f(\xi)\bigr\}g(\xi)\,d\xi$$

and using Cauchy–Schwarz

$$F'(\theta)\leqslant F(\theta)^{1/2}F''(\theta)^{1/2}.$$

Let $G(\theta)=F'(\theta)/F(\theta)$ with $\alpha<\theta<\beta$, then

$$G'(\theta)=\frac{F(\theta)F''(\theta)-F'(\theta)^2}{F(\theta)^2}\geqslant 0.$$

Hence $G(\theta)$ is nondecreasing. Now, we define $H(\theta)=\log F(\theta)$, and using the mean value theorem, we get

$$\log F(\theta_1)-\log F(\theta_2)=(\theta_1-\theta_2)G\bigl(\gamma\theta_1+(1-\gamma)\theta_2\bigr),$$

where $\gamma\in(0,1)$, $\theta_1\in(\alpha,\beta)$ and $\theta_2\in(\alpha,\theta_1]$, therefore

$$(\theta_1-\theta_2)G(\theta_2)\leqslant \log F(\theta_1)-\log F(\theta_2)\leqslant(\theta_1-\theta_2)G(\theta_1),$$

and these inequalities imply (2.25) and (2.26).

Now observe that by (2.23), Lebesgue's Dominated Convergence Theorem implies that inequality (2.25) is also true when $\theta_1\in(\alpha,\beta)$ and $\theta_2=\alpha$, which yields the result. □

Corollary 2.2. *Let $v\in H^{k+1}(\mathbb{R})$, $l\geqslant 0$, $\|v\|^2_{\theta,l}=\int_{\mathbb{R}}(l+\xi^2)^\theta|\widehat{v}(\xi)|^2\,d\xi$, $\theta_1\in(0,k)$, $\theta_2\in[0,\theta_1]$ and $G^l_v(\theta)$ as defined in (1.20), then we have*

$$\|v\|^2_{\theta_1,l}\leqslant\|v\|^2_{\theta_2,l}\exp\bigl\{(\theta_1-\theta_2)G^l_v(\theta_1)\bigr\}. \tag{2.27}$$

If $l\geqslant 1$ we also have:

$$\|v\|^2_{\theta_1,l}\geqslant\|v\|^2_{\theta_2,l}\exp\bigl\{(\theta_1-\theta_2)G^l_v(\theta_2)\bigr\}. \tag{2.28}$$

If $0\leqslant l<1$ we also have (2.28) for $\theta_1\in(0,k)$, $\theta_2\in[c_0,\theta_1]$, where $c_0>0$.

Proof. The proof follows directly from Lemma 2.1, taking $f(\xi)=(l+\xi^2)$ and $g(\xi)=|\widehat{v}(\xi)|^2$ and from the inequality

$$|\log x|\leqslant\begin{cases}\frac{x^\sigma}{\sigma} & \text{if } x\geqslant 1,\\ \frac{x^{-\sigma}}{\sigma} & \text{if } 0<x\leqslant 1,\end{cases}$$

for all $\sigma>0$. In fact, for $j=0,1,2$ and $\sigma\in(0,1/2)$ constant we have when $l\geqslant 1$ and $\theta\in[0,1]$

$$\begin{aligned}\bigl(l+\xi^2\bigr)^\theta\bigl|\log\bigl(l+\xi^2\bigr)\bigr|^j\bigl|\widehat{v}(\xi)\bigr|^2&\leqslant\frac{l^2}{\sigma^2}\bigl(1+\xi^2\bigr)^{1+2\sigma}\bigl|\widehat{v}(\xi)\bigr|^2\\&\leqslant C(l)\bigl(1+\xi^2\bigr)^2\bigl|\widehat{v}(\xi)\bigr|^2\in L^1,\end{aligned}$$

and when $0\leqslant l<1$, $\theta\in[c_0,k]$ and $j=0,1,2$ we have

$$\begin{aligned}\bigl(l+\xi^2\bigr)^\theta\bigl|\log\bigl(l+\xi^2\bigr)\bigr|^j\bigl|\widehat{v}(\xi)\bigr|^2&\leqslant\begin{cases}(4^2/\theta^2)(l+\xi^2)^{\theta-j\theta/4}|\widehat{v}(\xi)|^2 & \text{if } |\xi|<\sqrt{1-l},\\ (1/\sigma^2)(l+\xi^2)^{1+\sigma j}|\widehat{v}(\xi)|^2 & \text{if } |\xi|\geqslant\sqrt{1-l},\end{cases}\\&\leqslant C(c_0)\bigl(1+\xi^2\bigr)^2\bigl|\widehat{v}(\xi)\bigr|^2\in L^1.\end{aligned} \tag{2.29}$$

For the proof of (2.27) when $0\leqslant l<1$, $\theta_1\in(0,k)$ and $\theta_2\in[0,\theta_1]$ we use (2.29) and

$$\lim_{\theta_2\to b^+}\|v\|_{\theta_2,l}=\|v\|_{b,l},\quad b\geqslant 0,$$

which is a consequence of Lebesgue's Dominated Convergence Theorem. □

Remark 2.3.

(1) Let $w \in H^{2k}$, then there exists $v \in H^{2k}$, such that

$$G_w(k) = -G_v(k) \quad \text{and} \quad \|w\|_{\dot{H}^k} = \|v\|_{\dot{H}^k}. \tag{2.30}$$

In fact, if v is defined by

$$\widehat{w}(x) = \frac{1}{x^{2k+1}} \widehat{v}\left(\frac{1}{x}\right),$$

then $\|w\|_{\dot{H}^\theta} = \|v\|_{\dot{H}^{2k-\theta}}$ and therefore satisfies (2.30).

(2) Let $v \in H^{k+1}(\mathbb{R})$, $l \geqslant 0$, Corollary 2.2 and the fact that $G_v^l(\theta)$ is nondecreasing implies the following:
If $0 \leqslant \theta_2 \leqslant \theta_1 \leqslant \theta_0 < k$ and $G_v^l(\theta_0) \leqslant 0$, then $\|v\|_{\theta_1,l} \leqslant \|v\|_{\theta_2,l}$.
If $0 < \theta_0 \leqslant \theta_2 \leqslant \theta_1 < k$ and $G_v^l(\theta_0) \geqslant 0$, then $\|v\|_{\theta_2,l} \leqslant \|v\|_{\theta_1,l}$.

(3) Let $v \in H^{k+1}(\mathbb{R})$ and $0 < \theta_2 < \theta_1 < \theta_3 < k$, by interpolation we know that

$$\|v\|_{H^{\theta_1}} \leqslant \|v\|_{H^{\theta_2}}^{\beta} \|v\|_{H^{\theta_3}}^{1-\beta},$$

where $\beta = (\theta_3 - \theta_1)/(\theta_3 - \theta_2)$, hence

$$\left(\frac{\|v\|_{H^{\theta_1}}}{\|v\|_{H^{\theta_2}}}\right)^{1/(\theta_1-\theta_2)} \leqslant \left(\frac{\|v\|_{H^{\theta_3}}}{\|v\|_{H^{\theta_1}}}\right)^{1/(\theta_3-\theta_1)}.$$

On the other hand, the inequalities (2.27) and (2.28) give

$$\left(\frac{\|v\|_{H^{\theta_1}}}{\|v\|_{H^{\theta_2}}}\right)^{1/(\theta_1-\theta_2)} \leqslant e^{\frac{\|v\|_{\theta_1,1}^2}{2\|v\|_{H^{\theta_1}}^2}} \leqslant \left(\frac{\|v\|_{H^{\theta_3}}}{\|v\|_{H^{\theta_1}}}\right)^{1/(\theta_3-\theta_1)}.$$

3. Estimates of the norms for the complex mKdV equation and Benjamin–Ono equation

The proof of Theorem 1.3 and the proof of Theorem 1.4, are consequence from Theorem 1.2 and from the propositions to follow.

The following elementary lemma will be useful in the proof of Propositions 3.2 and 3.5.

Lemma 3.1. *Let $x \geqslant 0$, $a \geqslant 0$ and $b \geqslant 0$, if $x^2 \leqslant ax + b$, then*

$$0 \leqslant x \leqslant a + b^{1/2}.$$

Proposition 3.2. *Let $u_0 \in H^1(\mathbb{R})$ and $u(t)$ be the solution of IVP* (1.4)*, then*

$$\big| \|u_x(t)\|_{L_x^2} - \|u_x(0)\|_{L^2} \big| \leqslant c|k_0| \|u(0)\|_{L^2}^3,$$

where $k_0 = -(e+d)/(6b)$.

Proof. If $k_0 < 0$ in (1.6), Gagliardo–Nirenberg's inequality yields

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u(x,t)|^2\,dx &\leqslant |k_0| \int_{\mathbb{R}} |u(x,t)|^4\,dx + \int_{\mathbb{R}} |\partial_x u(x,0)|^2\,dx \\ &\leqslant c|k_0| \|u(0)\|_{L^2}^3 \|u_x(t)\|_{L_x^2} + \int_{\mathbb{R}} |\partial_x u(x,0)|^2\,dx,\end{aligned}$$

therefore by Lemma 3.1 we have

$$\|u_x(t)\|_{L_x^2} \leqslant \|u_x(0)\|_{L^2} + c|k_0| \|u(0)\|_{L^2}^3.$$

If $k_0 > 0$ in (1.6), applying Gagliardo–Nirenberg's inequality we have the estimate

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u(x,t)|^2\,dx &\leqslant k_0 \int_{\mathbb{R}} |u(x,0)|^4\,dx + \int_{\mathbb{R}} |\partial_x u(x,0)|^2\,dx \\ &\leqslant ck_0 \|u(0)\|_{L^2}^3 \|u_x(0)\|_{L^2} + \int_{\mathbb{R}} |\partial_x u(x,0)|^2\,dx.\end{aligned} \tag{3.31}$$

Similarly, we have

$$\int_{\mathbb{R}} \left|\partial_x u(x,0)\right|^2 dx \leqslant k_0 \int_{\mathbb{R}} \left|u(x,t)\right|^4 dx + \int_{\mathbb{R}} \left|\partial_x u(x,t)\right|^2 dx$$
$$\leqslant ck_0 \left\|u(0)\right\|_{L^2}^3 \left\|u_x(t)\right\|_{L_x^2} + \int_{\mathbb{R}} \left|\partial_x u(x,t)\right|^2 dx,$$

therefore

$$\begin{aligned}\left\|u_x(0)\right\|_{L^2} &\leqslant c^{1/2} k_0^{1/2} \left\|u(0)\right\|_{L^2}^{3/2} \left\|u_x(t)\right\|_{L_x^2}^{1/2} + \left\|u_x(t)\right\|_{L_x^2}\\ &\leqslant \frac{ck_0}{2} \left\|u(0)\right\|_{L^2}^3 + \frac{1}{2}\left\|u_x(t)\right\|_{L_x^2} + \left\|u_x(t)\right\|_{L_x^2},\end{aligned} \tag{3.32}$$

using (3.32) in (3.31) it follows that

$$\left\|u_x(t)\right\|_{L_x^2}^2 \leqslant ck_0 \left\|u(0)\right\|_{L^2}^3 \left(\frac{ck_0}{2}\left\|u(0)\right\|_{L^2}^3 + \frac{3}{2}\left\|u_x(t)\right\|_{L_x^2}\right) + \left\|u_x(0)\right\|_{L^2}^2, \tag{3.33}$$

and using again Lemma 3.1 we obtain

$$\left\|u_x(t)\right\|_{L_x^2} \leqslant \left\|u_x(0)\right\|_{L^2} + ck_0 \left\|u(0)\right\|_{L^2}^3.$$

Analogously, by symmetry we have

$$\left\|u_x(0)\right\|_{L^2} \leqslant \left\|u_x(t)\right\|_{L_x^2} + c|k_0| \left\|u(0)\right\|_{L^2}^3.$$

This concludes the proof of the proposition. □

Corollary 3.3. *Let $u_0 \in H^1(\mathbb{R})$ and $u(t)$ the solution of IVP* (1.4)*, then*

$$\left|\left\|u(t)\right\|_{H^1} - \left\|u(0)\right\|_{H^1}\right| \leqslant \left\|u(0)\right\|_{L^2} + c|k_0| \left\|u(0)\right\|_{L^2}^3. \tag{3.34}$$

Proof. Let us $P = \|u(0)\|_{L^2}$, $Q(t) = \|u_x(t)\|_{L_x^2}$ and $R = \|u(t)\|_{H^1} - \|u(0)\|_{H^1}$, we have

$$R + Q(0) \leqslant R + \left(P^2 + Q(0)^2\right)^{1/2} = \left(P^2 + Q(t)^2\right)^{1/2} \leqslant P + Q(t),$$

and therefore by Proposition 3.2 we get

$$R \leqslant P + Q(t) - Q(0) \leqslant P + c|k_0|P^3. \qquad \square$$

Remark 3.4.

(1) Corollary 3.3 implies Proposition 3.2. In fact, let $\lambda > 0$ a constant and $\tilde{u}(x,t) = \lambda u(\lambda x, \lambda^3 t)$ the scaling solution of IVP (1.4), we have

$$\left(\int_{\mathbb{R}} \left(1+\xi^2\right)^{\theta} \left|\widehat{\tilde{u}}(t)\right|^2 d\xi\right)^{1/2} = \lambda^{\theta+1/2} \left(\int_{\mathbb{R}} \left(\frac{1}{\lambda^2} + \xi^2\right)^{\theta} \left|\widehat{u}\left(\lambda^3 t\right)\right|^2 d\xi\right)^{1/2}. \tag{3.35}$$

By (3.35) and Corollary 3.3 for $\tilde{u}(x,t)$, we obtain for all $\lambda > 0$

$$\left|\left(\int_{\mathbb{R}} \left(\frac{1}{\lambda^2} + \xi^2\right) \left|\widehat{u}(t)\right|^2 d\xi\right)^{1/2} - \left(\int_{\mathbb{R}} \left(\frac{1}{\lambda^2} + \xi^2\right) \left|\widehat{u}(0)\right|^2 d\xi\right)^{1/2}\right| \leqslant \frac{1}{\lambda} P + cP^3,$$

where $P = \|u(0)\|_{L^2}$, therefore

$$\left|\left\|u_x(t)\right\|_{L^2} - \left\|u_x(0)\right\|_{L^2}\right| \leqslant c\left\|u(0)\right\|_{L^2}^3.$$

(2) Note that $\|u(t)\|_{H^\theta}^2 \leqslant \|u(0)\|_{H^\theta}^2 + A^2$ implies $\|u(t)\|_{H^\theta} \leqslant \|u(0)\|_{H^\theta} + A$, however $\|u(t)\|_{H^\theta} \leqslant \|u(0)\|_{H^\theta} + A$ no implies $\|u(t)\|_{H^\theta}^2 \leqslant \|u(0)\|_{H^\theta}^2 + g(A)$ for some function g independent of $u(0)$ and $u(t)$ (see (3.36) and (4.58)).

(3) If $e^{G(1)} \leqslant \|u(0)\|_{L^2}^4$ then

$$\left\|u_x(t)\right\|_{L^2}^2 \leqslant \left\|u_x(0)\right\|_{L^2}^2 + c\left\|u(0)\right\|_{L^2}^6. \tag{3.36}$$

In fact let $Q(t) = \|D_x u(t)\|_{L^2}$. From (1.6), Corollary 2.2, Proposition 3.2 and the Gagliardo–Nirenberg's inequality, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} u^4(x,t)\,dx &\leqslant \int_{\mathbb{R}} u^4(x,0) + c\big|\big\|Du(0)\big\|_{L^2}^2 - \big\|D_x u(t)\big\|_{L^2_x}^2\big| \\
&\leqslant cQ(0)\big\|u(0)\big\|_{L^2}^3 + c\big|Q(0) - Q(t)\big|\big(Q(0) + Q(t)\big) \\
&\leqslant cQ(0)\big\|u(0)\big\|_{L^2}^3 + c\big\|u(0)\big\|_{L^2}^3\big(2Q(0) + \big\|u(0)\big\|_{L^2}^3\big) \\
&\leqslant cQ(0)\big\|u(0)\big\|_{L^2}^3 + c\big\|u(0)\big\|_{L^2}^6 \\
&\leqslant ce^{G(1)/2}\big\|u(0)\big\|_{L^2}^4 + c\big\|u(0)\big\|_{L^2}^6 \\
&\leqslant c\big\|u(0)\big\|_{L^2}^4\big(e^{G(1)/2} + \big\|u(0)\big\|_{L^2}^2\big) \\
&\leqslant c\big\|u(0)\big\|_{L^2}^6,
\end{aligned} \tag{3.37}$$

and by conserved quantity (1.6) we obtain the result.

As in the previous proposition we also have

Proposition 3.5. *Let $u_0 \in H^{1/2}(\mathbb{R})$ and $u(t)$ be the solution of IVP (1.7), then*

$$\big|\big\|D_x^{1/2}u(t)\big\|_{L^2} - \big\|D_x^{1/2}u(0)\big\|_{L^2}\big| \leqslant c|k_0|\big\|u(0)\big\|_{L^2}^2. \tag{3.38}$$

Proof. The Gagliardo–Nirenberg inequality

$$\big\|u(t)\big\|_{L^3}^3 \leqslant c\big\|D_x^{1/2}u(t)\big\|_{L^2}\big\|u(t)\big\|_{L^2}^2,$$

and the conservations laws (1.8) and (1.9) imply

$$\big\|D_x^{1/2}u(t)\big\|_{L^2}^2 \leqslant \big\|D_x^{1/2}u(0)\big\|_{L^2}^2 + c|k_0|\big\|D_x^{1/2}u(t)\big\|_{L^2}\big\|u(0)\big\|_{L^2}^2 + c|k_0|\big\|D_x^{1/2}u(0)\big\|_{L^2}\big\|u(0)\big\|_{L^2}^2 \tag{3.39}$$

and

$$\begin{aligned}
\big\|D_x^{1/2}u(0)\big\|_{L^2}^2 &\leqslant \big\|D_x^{1/2}u(t)\big\|_{L^2}^2 + c|k_0|\big\|D_x^{1/2}u(t)\big\|_{L^2}\big\|u(0)\big\|_{L^2}^2 + c|k_0|\big\|D_x^{1/2}u(0)\big\|_{L^2}\big\|u(0)\big\|_{L^2}^2 \\
&\leqslant \frac{3}{2}\big\|D_x^{1/2}u(t)\big\|_{L^2}^2 + c|k_0|\big\|u(0)\big\|_{L^2}^4 + \frac{1}{2}\big\|D_x^{1/2}u(0)\big\|_{L^2}^2,
\end{aligned}$$

thus

$$\big\|D_x^{1/2}u(0)\big\|_{L^2} \leqslant \sqrt{3}\big\|D_x^{1/2}u(t)\big\|_{L^2} + c|k_0|\big\|u(0)\big\|_{L^2}^2, \tag{3.40}$$

combining (3.39) and (3.40) we have

$$\big\|D_x^{1/2}u(t)\big\|_{L^2}^2 \leqslant \big\|D_x^{1/2}u(0)\big\|_{L^2}^2 + c|k_0|\big\|D_x^{1/2}u(t)\big\|_{L^2}\big\|u(0)\big\|_{L^2}^2 + c|k_0|^2\big\|u(0)\big\|_{L^2}^4,$$

and Lemma 3.1 gives

$$\big\|D_x^{1/2}u(t)\big\|_{L^2} \leqslant \big\|D_x^{1/2}u(0)\big\|_{L^2} + c|k_0|\big\|u(0)\big\|_{L^2}^2.$$

Similarly we have

$$\big\|D_x^{1/2}u(0)\big\|_{L^2} \leqslant \big\|D_x^{1/2}u(t)\big\|_{L^2} + c|k_0|\big\|u(0)\big\|_{L^2}^2,$$

and the proposition follows. □

4. Preliminary estimates

In this section we will prove Theorem 1.3.

Lemma 4.1. *Let $q > 1$, $\delta_0 \in [0,1]$ and $\gamma_0 = \delta_0 q/(e\log q)$, then for all $x > 0$ we have*

$$\begin{aligned}
x\log q &\leqslant \frac{q}{e}(1-\delta_0) + (x+\gamma_0)\log(x+\gamma_0) \\
&\leqslant (x+\gamma_0+\gamma_1)\log(x+\gamma_0+\gamma_1),
\end{aligned} \tag{4.41}$$

where $\gamma_1 = \max\{q(1-\delta_0)/e, e\} \leqslant qe$. In particular, when $\delta_0 = 1$

$$x\log q \leqslant (x+\gamma_0)\log(x+\gamma_0). \tag{4.42}$$

Proof. The proof is immediate because the function

$$f(x) = (x+\gamma_0)\log(x+\gamma_0) - x\log q$$

has a minimum at $x_{\min} = q/e - \gamma_0$ and

$$f(x_{\min}) = -\frac{q}{e} + \gamma_0 \log q = \frac{q}{e}(\delta_0 - 1). \qquad \square$$

Proposition 4.2. *Let $u_0 \in H^{k+1}(\mathbb{R})$ and let $u(t)$ be the solution of IVP (1.11) with initial data u_0, then*

$$\|u(t)\|_{\dot{H}^\theta} \leqslant \|u_0\|_{L^2} e^{\theta G(k)/2} + c \cdot |k_0|^{\theta\Psi/k} \|u_0\|_{L^2}^{\sigma}, \tag{4.43}$$

where $\sigma = 1 + (n-1)\theta/k$.

Proof. By Corollary 2.2 we have

$$\|u_0\|_{\dot{H}^k} \leqslant e^{kG(k)/2} \|u_0\|_{L^2},$$

and by (1.12) is

$$\begin{aligned} \|u(t)\|_{\dot{H}^k} &\leqslant \|u_0\|_{\dot{H}^k} + c \cdot |k_0|^{\Psi} \|u_0\|_{L^2}^n \\ &\leqslant e^{kG(k)/2}\|u_0\|_{L^2} + c \cdot |k_0|^{\Psi} \|u_0\|_{L^2}^n. \end{aligned} \tag{4.44}$$

Interpolation and (4.44) gives

$$\begin{aligned} \|u(t)\|_{\dot{H}^\theta} &\leqslant \|u(t)\|_{\dot{H}^k}^{\theta/k} \|u_0\|_{L^2}^{1-\theta/k} \\ &\leqslant \big(e^{\theta G(k)/2}\|u_0\|_{L^2}^{\theta/k} + c \cdot |k_0|^{\theta\Psi/k} \|u_0\|_{L^2}^{n\theta/k}\big)\|u_0\|_{L^2}^{1-\theta/k} \\ &\leqslant \|u_0\|_{L^2} e^{\theta G(k)/2} + c \cdot |k_0|^{\theta\Psi/k}\|u_0\|_{L^2}^{\sigma}. \qquad \square \end{aligned}$$

Corollary 4.3. *Let $u_0 \in H^{k+1}(\mathbb{R})$ and let $u(t)$ be the solution of IVP (1.4) with initial data u_0, if $G(k) \leqslant 0$, then*

$$\|u(t)\|_{\dot{H}^\theta} \leqslant \|u_0\|_{L^2} + c \cdot |k_0|^{\theta\Psi/k} \|u_0\|_{L^2}^{1+(n-1)\theta/k}. \tag{4.45}$$

Proof. The proof follows directly of Proposition 4.2. $\square$

Corollary 4.4. *Let $u_0 \in H^k(\mathbb{R})$ and let $u(t)$ be the solution of IVP (1.4) with initial data u_0. If $e^{G(k)} \leqslant \|u_0\|_{L^2}^{2(n-1)/k}$ then*

$$\|u(t)\|_{\dot{H}^\theta} \leqslant c\big(1 + |k_0|^{\theta\Psi/k}\big)\|u_0\|_{L^2}^{1+(n-1)\theta/k}. \tag{4.46}$$

Proof. The proof also follows directly of Proposition 4.2. $\square$

In the next we will find some estimates in order to obtain a result better (Lemma 4.7) than the obtained one in Corollary 4.4 (see Remark 4.6(4)).

For $t > 0$ we consider the functions

$$\varphi_t : [0,k] \to \mathbb{R}; \quad \delta \mapsto \varphi_t(\delta) := \|u(t)\|_{\dot{H}^\delta} - \kappa_\alpha^n \|u(0)\|_{\dot{H}^\delta}, \tag{4.47}$$

and

$$\phi_t : [0,k] \to \mathbb{R}; \quad \delta \mapsto \phi_t(\delta) := \|u(t)\|_{H^\delta} - \kappa_\alpha^n \|u(0)\|_{H^\delta}, \tag{4.48}$$

where $\kappa_\alpha^n = (n-\alpha)/(1-\alpha)$, $\alpha < 1$ and we define δ_1 and δ_0 such that

$$\varphi_t(\delta_1) = \max_{\delta\in[0,k]} \varphi_t(\delta), \qquad \phi_t(\delta_0) = \max_{\delta\in[0,k]} \phi_t(\delta). \tag{4.49}$$

We have the following theorem.

Theorem 4.5. *Let $u \in \mathcal{C}(\mathbb{R}, H^{k+1}(\mathbb{R}))$, such that for all $t \in \mathbb{R}$, $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$. If $\delta_1 = \delta_1(\varphi_t) \in (0,k)$, then for all $\theta \in [0,k]$, $\zeta > 0$, $n > 0$ and $\alpha < 1$ we have*

$$\begin{aligned} \|u(t)\|_{\dot{H}^\theta} \leqslant{}& \kappa_\alpha^n \|u(0)\|_{\dot{H}^\theta} + c_\alpha^n \zeta \|u_0\|_{L^2}^n \left(\left(\frac{1}{|G(\delta_1)|} + 1 \right) \exp\left\{ \frac{\kappa_\alpha^n \delta_1}{2} G(\delta_1) \right\} + 1 \right) \\ &+ c_\alpha^n \frac{\|u_0\|_{L^2}^{\alpha}}{\zeta^{(1-\alpha)/(n-1)} \log(\frac{1}{\zeta^{1/(n-1)}\|u_0\|_{L^2}})} \chi_{\{\zeta^{1/(n-1)}\|u_0\|_{L^2} < 1\}}, \end{aligned} \tag{4.50}$$

where $c_\alpha^n \lesssim \kappa_\alpha^n$ and $G(\delta_1)$ was defined in (1.21).

If $\delta_0=\delta_0(\phi_t)\in(0,k)$, then for all $\theta\in[0,k]$ we have

$$\|u(t)\|_{H^\theta}\leqslant\|u(0)\|_{H^\theta}+\|u_0\|_{L^2}\left(\frac{2\|u(0)\|^2_{H^{\delta_0}}}{e|\!|\!|u(0)|\!|\!|^2_{\delta_0}}+e\right)\exp\left\{\frac{\delta_0}{2}\frac{|\!|\!|u(0)|\!|\!|^2_{\delta_0}}{\|u(0)\|^2_{H^{\delta_0}}}\right\}. \tag{4.51}$$

Proof. Without loss of generality we will prove the theorem for $n=3$ and $k=1$. The same argument provides the theorem in the general case.

If $\delta_1\in(0,1)$ then $\varphi_t'(\delta_1)=0$, therefore

$$\frac{\int\xi^{2\delta_1}\log(\xi^2)|\widehat{u}(t,\xi)|^2\,d\xi}{\|u(t)\|_{\dot H^{\delta_1}}}=\frac{\kappa_\alpha\int\xi^{2\delta_1}\log(\xi^2)|\widehat{u}(0,\xi)|^2\,d\xi}{\|u(0)\|_{\dot H^{\delta_1}}}. \tag{4.52}$$

From (2.27) and (4.52) we have

$$\begin{aligned}\|u(t)\|_{\dot H^{\delta_1}}&\leqslant\|u(0)\|_{L^2}\exp\left\{\frac{\delta_1\int\xi^{2\delta_1}\log(\xi^2)|\widehat{u}(t,\xi)|^2\,d\xi}{2\|u(t)\|^2_{\dot H^{\delta_1}}}\right\}\\&=\|u(0)\|_{L^2}\exp\left\{\frac{\kappa_\alpha\delta_1\int\xi^{2\delta_1}\log(\xi^2)|\widehat{u}(0,\xi)|^2\,d\xi}{2\|u(t)\|_{\dot H^{\delta_1}}\|u(0)\|_{\dot H^{\delta_1}}}\right\}.\end{aligned} \tag{4.53}$$

Let $\Upsilon(t)=\|u(t)\|_{\dot H^{\delta_1}}$ and $\Gamma=\|u(0)\|_{L^2}$, from (4.53) we obtain

$$\frac{\Upsilon(t)}{\zeta\Gamma^3}\leqslant\frac{1}{\zeta\Gamma^2}\exp\left\{\frac{\kappa_\alpha\delta_1}{2}\frac{\Lambda_{\delta_1,0}(u(0))}{\Upsilon(t)\Upsilon(0)}\right\}, \tag{4.54}$$

taking the logarithm in the inequality above and multiplying by $\Upsilon(t)/(\zeta\Gamma^3)$ it follows that

$$\begin{aligned}\frac{\Upsilon(t)}{\zeta\Gamma^3}\log\frac{\Upsilon(t)}{\zeta\Gamma^3}&\leqslant\frac{\Upsilon(0)}{\zeta\Gamma^3}\left(\frac{\kappa_\alpha\delta_1\Lambda_{\delta_1,0}(u(0))}{2\Upsilon(0)^2}\right)+\frac{2}{p}\frac{\Upsilon(t)}{\zeta\Gamma^3}\log\left(\frac{1}{\sqrt{\zeta}\Gamma}\right)^p\\&\leqslant\frac{\Upsilon(0)}{\zeta\Gamma^3}\log\tilde q+\frac{2}{p}\frac{\Upsilon(t)}{\zeta\Gamma^3}\log\left(\frac{1}{\sqrt{\zeta}\Gamma}\right)^p,\end{aligned} \tag{4.55}$$

where $p>2$ and

$$\log\tilde q=\frac{\kappa_\alpha\delta_1}{2}\frac{\Lambda_{\delta_1,0}(u(0))}{\Upsilon(0)^2}.$$

Here we consider two cases:

(1) If $\frac{\Upsilon(t)}{\zeta\Gamma^3}\leqslant1$, then $\|u(t)\|_{\dot H^{\delta_1}}-\|u(0)\|_{\dot H^{\delta_1}}\leqslant\zeta\|u(0)\|^3_{L^2}$ and in this case we obtain (4.50).

(2) If $\frac{\Upsilon(t)}{\zeta\Gamma^3}>1$, in this case we consider two sub-cases:

(2.1) If $\sqrt{\zeta}\Gamma<1$, it is clear from (4.54) that if $\Lambda_{\delta_1,0}(u(0))\leqslant0$ then for all $\alpha<1$

$$\Upsilon(t)\leqslant\Gamma\leqslant\frac{\Gamma^\alpha}{(1-\alpha)\zeta^{(1-\alpha)/2}\log(\frac{1}{\sqrt{\zeta}\Gamma})},$$

because that $\log x\leqslant x^{1-\alpha}/(1-\alpha)$ with $x=1/(\sqrt{\zeta}\Gamma)>1$, hence we obtain (4.50). Now, if $\Lambda_{\delta_1,0}(u(0))>0$ ($\tilde q>1$), Lemma 4.1 and (4.55) give

$$\begin{aligned}\frac{\Upsilon(t)}{\zeta\Gamma^3}\log\frac{\Upsilon(t)}{\zeta\Gamma^3}&\leqslant\left(\frac{\Upsilon(0)}{\zeta\Gamma^3}+\tilde k_0\right)\log\left(\frac{\Upsilon(0)}{\zeta\Gamma^3}+\tilde k_0\right)+\left(\frac{2}{p}\frac{\Upsilon(t)}{\zeta\Gamma^3}+\tilde k_1\right)\log\left(\frac{2}{p}\frac{\Upsilon(t)}{\zeta\Gamma^3}+\tilde k_1\right)\\&\leqslant\left(\frac{\Upsilon(0)}{\zeta\Gamma^3}+\tilde k_0+\frac{2}{p}\frac{\Upsilon(t)}{\zeta\Gamma^3}+\tilde k_1\right)\log\left(\frac{\Upsilon(0)}{\zeta\Gamma^3}+\tilde k_0+\frac{2}{p}\frac{\Upsilon(t)}{\zeta\Gamma^3}+\tilde k_1\right),\end{aligned} \tag{4.56}$$

where

$$\tilde k_0=\frac{\delta_1\tilde q}{e\log\tilde q}+\tilde q e\quad\text{and}\quad\tilde k_1=\frac{(\frac{1}{\sqrt{\zeta}\Gamma})^p}{e\log(\frac{1}{\sqrt{\zeta}\Gamma})^p}.$$

Since the function $g(x)=x\log x$ is nondecreasing for $x>1$, the inequality (4.56) implies

$$\begin{aligned}\|u(t)\|_{\dot H^\theta}\leqslant{}&\frac{p}{p-2}\|u(0)\|_{\dot H^\theta}+c_p\zeta\|u_0\|^3_{L^2}\left(\frac{\|u(0)\|^2_{\dot H^{\delta_1}}}{\Lambda_{\delta_1,0}(u(0))}+1\right)\exp\left\{\frac{\kappa_\alpha\delta_1}{2}\frac{\Lambda_{\delta_1,0}(u(0))}{\|u(0)\|^2_{\dot H^{\delta_1}}}\right\}\\&+c_p\frac{\Gamma^{3-p}}{\zeta^{(p-2)/2}\log(\frac{1}{\sqrt{\zeta}\|u_0\|_{L^2}})},\end{aligned}$$

where $c_p\lesssim p/(p-2)$ and consequently in this case we also have (4.50) with $p=3-\alpha$.

(2.2) If $\sqrt{\zeta}\Gamma \geqslant 1$, from (4.55) we have

$$\frac{\Upsilon(t)}{\zeta\Gamma^3}\log\frac{\Upsilon(t)}{\zeta\Gamma^3} \leqslant \frac{\Upsilon(0)}{\zeta\Gamma^3}\log\tilde{q},$$

thus in this case is $\log\tilde{q} > 0$ and therefore $\tilde{q} > 1$, then we use (4.41) to obtain

$$\|u(t)\|_{\dot{H}^\theta} \leqslant \|u(0)\|_{\dot{H}^\theta} + c\zeta\|u_0\|_{L^2}^3\left(\frac{\|u(0)\|_{\dot{H}^{\delta_1}}^2}{\Lambda_{\delta_1,0}(u(0))}+1\right)\exp\left\{\frac{\kappa_\alpha\delta_1}{2}\frac{\Lambda_{\delta_1,0}(u(0))}{\|u(0)\|_{\dot{H}^{\delta_1}}^2}\right\},$$

and accordingly Eq. (4.50).

The proof of inequality (4.51) is not difficult since $|||u(0)|||_{\delta_0} > 0$. In fact, for t fixed, we consider the function ϕ_t defined in (4.48) and let δ_0 such that

$$\phi_t(\delta_0) = \max_{\delta\in[0,1]}\phi_t(\delta).$$

If $\delta_0 \in (0,1)$ as above we obtain

$$\frac{\tilde{\Upsilon}(t)}{\Gamma} \leqslant \exp\left\{\frac{\delta_0}{2}\frac{|||u(0)|||_{\delta_0}^2}{\tilde{\Upsilon}(t)\tilde{\Upsilon}(0)}\right\},$$

where $\tilde{\Upsilon}(t) = \|u(t)\|_{H^{\delta_0}}$. Taking logarithm in the inequality above and multiplying by $\tilde{\Upsilon}(t)/\Gamma$ we have

$$\begin{aligned}\frac{\tilde{\Upsilon}(t)}{\Gamma}\log\frac{\tilde{\Upsilon}(t)}{\Gamma} &\leqslant \frac{\delta_0}{2}\frac{|||u(0)|||_{\delta_0}^2}{\tilde{\Upsilon}(0)\Gamma}\\ &= \frac{\tilde{\Upsilon}(0)}{\Gamma}\left(\frac{\delta_0}{2}\frac{|||u(0)|||_{\delta_0}^2}{\tilde{\Upsilon}(0)^2}\right)\\ &= \frac{\tilde{\Upsilon}(0)}{\Gamma}\log q,\end{aligned}$$

where

$$q = \exp\left\{\frac{\delta_0}{2}\frac{|||u(0)|||_{\delta_0}^2}{\tilde{\Upsilon}(0)^2}\right\} > 1,$$

and Lemma 4.1 (inequality (4.41)) implies inequality (4.51). This completes the proof. □

Remark 4.6.

(1) If in the proof of Theorem 4.5 we use inequality (4.42), then we get

$$\|u(t)\|_{H^\theta} \leqslant \|u(0)\|_{H^\theta} + \|u_0\|_{L^2}\left(\frac{2\|u(0)\|_{H^{\delta_0}}^2}{e\delta_0|||u(0)|||_{\delta_0}^2}\right)\exp\left\{\frac{\delta_0}{2}\frac{|||u(0)|||_{\delta_0}^2}{\|u(0)\|_{H^{\delta_0}}^2}\right\}. \tag{4.57}$$

(2) Let $u_0 \in L^2(\mathbb{R})$, $N > 0$, $v_0(x) = (\chi_{\{|\xi|<N\}}\widehat{u}_0)^\vee(x)$ and $v(t)$ be a solution of IVP (1.4) with initial data v_0, then we also have the following estimate, which is more refined than (4.51) and (4.57) (see Remark 3.4(2)):

$$\|v(t)\|_{H^\theta}^2 \leqslant \|v(0)\|_{H^\theta}^2 + c\langle N\rangle^2 Q\left(\|v(0)\|_{L^2}\right), \tag{4.58}$$

where $Q(x) = x^2 + x^6$ and $\theta \in [0,1]$. A proof of this inequality is given in Appendix A.

(3) Note that in inequality (4.51) and (4.57) we have

$$\frac{\|u(0)\|_{H^{\delta_0}}^2}{|||u(0)|||_{\delta_0}^2} \leqslant \frac{\|u(0)\|_{L^2}^2}{|||u(0)|||_0^2},$$

since the function $G(\theta)$, $\theta \in [0,k]$ in Lemma 2.1 is nondecreasing.

(4) If $e^{G(1)} \leqslant \|u_0\|_{L^2}^4$, Corollary 2.2 implies that

$$\|u_0\|_{\dot{H}^\theta} \leqslant e^{G(\theta)\theta/2}\|u_0\|_{L^2} \leqslant e^{G(1)\theta/2}\|u_0\|_{L^2} \leqslant \|u_0\|_{L^2}^{1+2\theta}.$$

Therefore the following lemma is more fine than Corollary 4.4.

Lemma 4.7. *Let $u_0 \in H^{k+1}(\mathbb{R})$, $\mu \geqslant 1$ and $u(t)$ be the solution of some differential equation satisfying* (1.11) *and* (1.12) *with initial data u_0 and with scaling* (1.2) *and* (1.3). *If $e^{G(k)} \leqslant |\mu k_0|^{2\Psi/k}\|u_0\|_{L^2}^{2(n-1)/k}$ then for all $\theta \in [0,k]$ we have*

$$\|u(t)\|_{\dot{H}^\theta} \leqslant \|u_0\|_{\dot{H}^\theta} + c|\mu k_0|^{\Psi\theta/k}\|u_0\|_{L^2}^{1+(n-1)\theta/k}, \tag{4.59}$$

where $G(\theta)$ was defined in (1.21).

Proof. Initially we will prove the inequality

$$\|u(t)\|_{\dot H^\theta} \leqslant \kappa_\alpha^n \|u_0\|_{\dot H^\theta} + c\kappa_\alpha^n |\mu k_0|^{\Psi\theta/k} \|u_0\|_{L^2}^{1+(n-1)\theta/k}.$$

Now let

$$\lambda = \frac{1}{|2\mu k_0|^{\Psi/(\beta_2 k)} \|u_0\|_{L^2}^{(n-1)/(\beta_2 k)}},$$

β_1, $\beta_2 \neq 0$ as in (1.3) and $\tilde u(x,t) = \lambda^{\beta_1} u(\lambda^{\beta_2} x, \vartheta t)$ the scaling solution, note that

$$\lambda = \frac{1}{|2\mu k_0|^{\Psi/\{(n-1)r(0)\}} \|u_0\|_{L^2}^{1/r(0)}}, \quad \text{and} \quad r(0) = \beta_2 k/(n-1). \tag{4.60}$$

Similarly as above, we consider the function

$$\tilde\varphi_t : [0,k] \to \mathbb{R}; \quad \delta \mapsto \tilde\varphi_t(\delta) := \|\tilde u(t/\vartheta)\|_{\dot H^\delta} - \kappa_\alpha^n \|\tilde u(0)\|_{\dot H^\delta}$$

and

$$\tilde\varphi_t(\delta_\lambda) = \max_{\delta\in[0,k]} \tilde\varphi_t(\delta). \tag{4.61}$$

If $\delta_\lambda = k$, then since that $\|\tilde u(t/\vartheta)\|_{\dot H^\theta} = \lambda^{r(\theta)} \|u(t)\|_{\dot H^\theta}$ and $r(\theta) = r(0) + \beta_2\theta$, from (1.12) and definition of δ_λ we have for all $\theta \in [0,k]$

$$\begin{aligned}
\|u(t)\|_{\dot H^\theta} - \kappa_\alpha^n \|u(0)\|_{\dot H^\theta} &\leqslant c|k_0|^\Psi \lambda^{nr(0)-r(\theta)} \|u(0)\|_{L^2}^n \\
&\leqslant c\frac{|k_0|^\Psi}{|2\mu k_0|^{\Psi\{(n-1)r(0)-\beta_2\theta\}/\{(n-1)r(0)\}}} \frac{\|u(0)\|_{L^2}^n}{\|u(0)\|_{L^2}^{\{(n-1)r(0)-\beta_2\theta\}/r(0)}} \\
&\leqslant c\frac{|\mu k_0|^{\Psi\theta/k}}{|2\mu|^\Psi} \|u(0)\|_{L^2}^{1+(n-1)\theta/k} \\
&\leqslant c|\mu k_0|^{\Psi\theta/k} \|u(0)\|_{L^2}^{1+(n-1)\theta/k}.
\end{aligned}$$

If $\delta_\lambda = 0$, then $\|u(t)\|_{\dot H^\theta} \leqslant \kappa_\alpha^n \|u(0)\|_{\dot H^\theta}$.

Therefore we consider $\delta_\lambda \in (0,k)$. By Theorem 4.5, inequality (4.50), we have for all $\theta \in [0,k]$, $\alpha < 1$ and $\mu \geqslant 1$

$$\begin{aligned}
\|\tilde u(t/\vartheta)\|_{\dot H^\theta} &\leqslant \kappa_\alpha^n \|\tilde u(0)\|_{\dot H^\theta} + c_\alpha^n |\mu k_0|^\Psi \|\tilde u_0\|_{L^2}^n \left(\left(\frac{\|\tilde u(0)\|_{\dot H^{\delta_\lambda}}^2}{|\Lambda_{\delta_\lambda,0}(\tilde u(0))|} + 1 \right) \exp\left\{ \frac{\kappa_\alpha^n \delta_\lambda}{2} \frac{\Lambda_{\delta_\lambda,0}(\tilde u(0))}{\|\tilde u(0)\|_{\dot H^{\delta_\lambda}}^2} \right\} + 1 \right) \\
&\quad + \frac{c_\alpha^n \|\tilde u_0\|_{L^2}^\alpha \chi_{\{|\mu k_0|^{\Psi/(n-1)} \|\tilde u_0\|_{L^2} < 1\}}}{|\mu k_0|^{\Psi(1-\alpha)/(n-1)} \log(\frac{1}{|\mu k_0|^{\Psi/(n-1)} \|\tilde u_0\|_{L^2}})},
\end{aligned} \tag{4.62}$$

where $c_\alpha^n \lesssim \kappa_\alpha^n$. Since

$$\frac{\Lambda_{\delta_\lambda,0}(\tilde u(0))}{\|\tilde u(0)\|_{\dot H^{\delta_\lambda}}^2} = \frac{\Lambda_{\delta_\lambda,0}(u(0))}{\|u(0)\|_{\dot H^{\delta_\lambda}}^2} + \log \lambda^{2\beta_2}, \tag{4.63}$$

inequality (4.62) gives for all $\theta \in [0,k]$

$$\begin{aligned}
\|u(t)\|_{\dot H^\theta} &\leqslant \kappa_\alpha^n \|u_0\|_{\dot H^\theta} \\
&\quad + c_\alpha^n |\mu k_0|^\Psi \lambda^{nr(0)-r(\theta)} \|u_0\|_{L^2}^n \left(\frac{1}{|G(\delta_\lambda) + \log\lambda^{2\beta_2}|} + 1 \right) \lambda^{\beta_2 \kappa_\alpha^n \delta_\lambda} \exp\left\{ \frac{\kappa_\alpha^n \delta_\lambda G(\delta_\lambda)}{2} \right\} \\
&\quad + c_\alpha^n |\mu k_0|^\Psi \lambda^{nr(0)-r(\theta)} \|u_0\|_{L^2}^n \\
&\quad + c_\alpha^n \lambda^{\alpha r(0)} \|u_0\|_{L^2}^\alpha \frac{\chi_{\{|\mu k_0|^{\Psi/(n-1)} \lambda^{r(0)} \|u_0\|_{L^2} < 1\}}}{|\mu k_0|^{\Psi(1-\alpha)/(n-1)} \lambda^{r(\theta)} \log(\frac{1}{|\mu k_0|^{\Psi/(n-1)} \lambda^{r(0)} \|u_0\|_{L^2}})}.
\end{aligned} \tag{4.64}$$

In the proof of Lemma 2.1 was proved that the function $G(\theta) = \frac{\Lambda_\theta(u_0)}{\|u_0\|_{\dot H^\theta}^2}$ is nondecreasing, thus we see that

$$G(\delta_\lambda) = \frac{\Lambda_{\delta_\lambda}(u_0)}{\|u_0\|_{\dot H^{\delta_\lambda}}^2} \leqslant G(k) \leqslant \log\{ |\mu k_0|^{2\Psi/k} \|u_0\|_{L^2}^{2(n-1)/k} \} = -\log\lambda^{2\beta_2} - \log 4^{\Psi/k} \leqslant -\log\lambda^{2\beta_2},$$

consequently

$$\left|G(\delta_\lambda)+\log\lambda^{2\beta_2}\right| = -\log\lambda^{2\beta_2} - G(\delta_\lambda) \geqslant -\log\lambda^{2\beta_2} - G(k) \geqslant \frac{\Psi \log 4}{k},$$

and

$$\begin{aligned}\lambda^{\beta_2\kappa_\alpha^n\delta_\lambda}\exp\left\{\frac{\kappa_\alpha^n\delta_\lambda G(\delta_\lambda)}{2}\right\} &= \left(\frac{\exp\{G(\delta_\lambda)\}}{|2\mu k_0|^{2\Psi/k}\|u_0\|_{L^2}^{2(n-1)/k}}\right)^{\kappa_\alpha^n\delta_\lambda/2}\\ &\leqslant \left(\frac{\exp\{G(k)\}}{|2\mu k_0|^{2\Psi/k}\|u_0\|_{L^2}^{2(n-1)/k}}\right)^{\kappa_\alpha^n\delta_\lambda/2}\\ &\leqslant 1.\end{aligned}$$

By (4.60) we have $|\mu k_0|^{\Psi/(n-1)}\lambda^{r(0)}\|u_0\|_{L^2} = 2^{-\Psi/(n-1)}$ and since $r(\theta)=r(0)+\beta_2\theta$ the last term in the inequality (4.64) is estimated in the following way

$$\begin{aligned}c_\alpha\|u_0\|_{L^2}^{\alpha}\frac{1}{|\mu k_0|^{\Psi(1-\alpha)/(n-1)}\lambda^{r(\theta)-\alpha r(0)}\log\{2^{\Psi/(n-1)}\}} &= c_\alpha\|u_0\|_{L^2}^{\alpha}\|u_0\|_{L^2}^{\{r(0)(1-\alpha)+\beta_2\theta\}/r(0)}\\ &\quad\times\frac{|2\mu k_0|^{\{r(0)(1-\alpha)+\beta_2\theta\}\Psi/\{(n-1)r(0)\}}}{|\mu k_0|^{\Psi(1-\alpha)/(n-1)}\log 2^{\Psi/(n-1)}}\\ &\lesssim c_\alpha|\mu k_0|^{\Psi\theta/k}\|u_0\|_{L^2}^{1+(n-1)\theta/k}.\end{aligned}$$

In consequence the inequality (4.64) implies that

$$\left\|u(t)\right\|_{\dot H^\theta} \leqslant \kappa_\alpha^n\|u_0\|_{\dot H^\theta} + c\kappa_\alpha^n|\mu k_0|^{\Psi\theta/k}\|u_0\|_{L^2}^{1+(n-1)\theta/k}.$$

Taking the limit when $\alpha\to-\infty$ in the above inequality, we conclude the proof of lemma. □

The following corollaries are easy extended for the models (1.11)–(1.12), we give details for the complex m-KdV.

Corollary 4.8. *Let $u_0\in H^2(\mathbb{R})$, if $\vartheta\in[0,1)$, $\eta\geqslant e^{-\vartheta G(1)/2}$, $e^{G(1)}\leqslant|\eta\tilde k_0|^{2/(1-\vartheta)}\|u_0\|_{L^2}^{4/(1-\vartheta)}$ and $u(t)$ be the solution of the complex m-KdV* (1.4) *with initial data u_0, then*

$$\left\|u(t)\right\|_{\dot H^\theta} \leqslant \|u_0\|_{\dot H^\theta} + c|\eta\tilde k_0|^{\theta/(1-\vartheta)}\|u_0\|_{L^2}^{1+2\theta/(1-\vartheta)},$$

where $\tilde k_0=-(e+d)/(6b)$.

Proof. Corollary 4.8 follows considering $\mu=\eta e^{\vartheta G(1)/2}\geqslant 1$, in Lemma 4.7. □

Corollary 4.9. *Let $u_0\in H^2(\mathbb{R})$ and $u(t)$ be the solution of the complex m-KdV* (1.4) *with initial data u_0, then*

$$\left\|u(t)\right\|_{\dot H^\theta} \leqslant \|u_0\|_{\dot H^\theta} + c\psi_1^\theta\|u_0\|_{L^2}^{1+2\theta}, \tag{4.65}$$

where $\psi_1=\max\{e^{G(1)}/(\|u_0\|_{L^2}^4|\tilde k_0|),|\tilde k_0|\}$ and $\tilde k_0=-(e+d)/(6b)$.

Proof. We consider two cases:

(1) If $\|u_0\|_{L^2}^4\tilde k_0^2\geqslant e^{G(1)}$, Lemma 4.7 with $\mu=1$ gives the result.

(2) If $\|u_0\|_{L^2}^4\tilde k_0^2\leqslant e^{G(1)}$, let $\mu=e^{G(1)}/(\|u_0\|_{L^2}^4\tilde k_0^2)\geqslant 1$, then $\mu^2\|u_0\|_{L^2}^4\tilde k_0^2\geqslant e^{G(1)}$ and Lemma 4.7 gives

$$\left\|u(t)\right\|_{\dot H^\theta} \leqslant \|u_0\|_{\dot H^\theta} + c\left(e^{G(1)}/\left(\|u_0\|_{L^2}^4|\tilde k_0|\right)\right)^\theta\|u_0\|_{L^2}^{1+2\theta}, \tag{4.66}$$

from this inequality we obtain (4.65). □

Observe that in Corollary 4.9 we have $\psi_1\to\infty$ when $\tilde k_0\to 0$. In Theorem 1.3 we obtain a best result when $|\tilde k_0|$ is small enough.

4.1. Proof of Theorem 1.1

To prove Theorem 1.1 we need the next lemmas.

Lemma 4.10. *Let $u(t) \in H^{k+1}(\mathbb{R})$ be solution of some differential equation that satisfies* (1.1)–(1.3)*, then for all $\theta \in [0,k]$*

$$\left\|u(t)\right\|_{\dot H^\theta} \leqslant \kappa_\alpha^n \|u_0\|_{\dot H^\theta} + c_\alpha^n e^{-|G(k)|(k-\theta)/2}\|u_0\|_{L^2}^n + c_\alpha^n \|u_0\|_{L^2}^\alpha \frac{e^{\frac{|G(k)|}{2}(\theta+k(1-\alpha)/(n-1))-k\alpha/\{2(n-1)\}}}{\log(e\frac{e^{|G(k)|}}{\|u_0\|_{L^2}^{2(n-1)/k}})}\Phi, \tag{4.67}$$

where $\Phi = \chi_{\{\exp\{|G(k)|+1\}>\|u_0\|_{L^2}^{2(n-1)/k}\}}$, $c_\alpha^n \lesssim \kappa_\alpha^n$ and $G(\theta)$ was defined in (1.21)*.*

Proof. As in the proof of Lemma 4.7, we use scaling with parameter

$$\lambda = e^{-\varsigma}, \quad \varsigma = \frac{|G(k)|+1}{2\beta_2},$$

we will prove this lemma for the complex mKdV equation (1.4), the proof for the general case is similar.

Let $\tilde u(x,t) = \lambda u(\lambda x, \lambda^3 t)$ be the scaling solution of IVP (1.4), $\tilde\varphi_t : [0,1] \to \mathbb{R}$; $\delta \mapsto \tilde\varphi_t(\delta) := \|\tilde u(t/\lambda^3)\|_{\dot H^\delta} - k_\alpha\|\tilde u(0)\|_{\dot H^\delta}$ and

$$\tilde\varphi_t(\delta_\lambda) = \max_{\delta\in[0,1]} \tilde\varphi_t(\delta). \tag{4.68}$$

If $\delta_\lambda = 1$, then since that $\|\tilde u(t/\lambda^3)\|_{\dot H^\theta} = \lambda^{\theta+1/2}\|u(t)\|_{\dot H^\theta}$, from Proposition 3.2 and definition of δ_λ we have for all $\theta \in [0,1]$

$$\left\|u(t)\right\|_{\dot H^\theta} - k_\alpha\left\|u(0)\right\|_{\dot H^\theta} \leqslant c\lambda^{1-\theta}\left\|u(0)\right\|_{L^2}^3 \leqslant ce^{-|G(1)|(1-\theta)/2}\left\|u(0)\right\|_{L^2}^3.$$

If $\delta_\lambda = 0$, then $\|u(t)\|_{\dot H^\theta} \leqslant k_\alpha\|u(0)\|_{\dot H^\theta}$.

Therefore we consider $\delta_\lambda \in (0,1)$. By Theorem 4.5, inequality (4.50), we have for all $\theta \in [0,1]$ and $\alpha < 1$

$$\begin{aligned}\left\|\tilde u\left(t/\lambda^3\right)\right\|_{\dot H^\theta} &\leqslant \kappa_\alpha\left\|\tilde u(0)\right\|_{\dot H^\theta} + c_\alpha\|\tilde u_0\|_{L^2}^3\left(\left(\frac{\|\tilde u(0)\|_{\dot H^{\delta_\lambda}}^2}{|\Lambda_{\delta_\lambda,0}(\tilde u(0))|}+1\right)\exp\left\{\frac{k_\alpha\delta_\lambda}{2}\frac{\Lambda_{\delta_\lambda,0}(\tilde u(0))}{\|\tilde u(0)\|_{\dot H^{\delta_\lambda}}^2}\right\}+1\right)\\ &\quad + c_\alpha\frac{\|\tilde u_0\|_{L^2}^\alpha\chi_{\{\|\tilde u_0\|_{L^2}<1\}}}{\log(\frac{1}{\|\tilde u_0\|_{L^2}})},\end{aligned} \tag{4.69}$$

where $c_\alpha \lesssim \kappa_\alpha$. Since

$$\frac{\Lambda_{\delta_\lambda,0}(\tilde u(0))}{\|\tilde u(0)\|_{\dot H^{\delta_\lambda}}^2} = \frac{\Lambda_{\delta_\lambda,0}(u(0))}{\|u(0)\|_{\dot H^{\delta_\lambda}}^2} + \log\lambda^2, \tag{4.70}$$

inequality (4.69) gives for all $\theta \in [0,1]$

$$\begin{aligned}\left\|u(t)\right\|_{\dot H^\theta} &\leqslant \kappa_\alpha\|u_0\|_{\dot H^\theta} + c_\alpha\lambda^{1-\theta}\|u_0\|_{L^2}^3\left(\frac{1}{|G(\delta_\lambda)+\log\lambda^2|}+1\right)\lambda^{k_\alpha\delta_\lambda}\exp\left\{\frac{k_\alpha\delta_\lambda}{2}\frac{\Lambda_{\delta_\lambda,0}(u_0)}{\|u_0\|_{\dot H^{\delta_\lambda}}^2}\right\}\\ &\quad + c_\alpha\lambda^{1-\theta}\|u_0\|_{L^2}^3 + c_\alpha\lambda^{\alpha/2}\|u_0\|_{L^2}^\alpha\frac{\chi_{\{\lambda^{1/2}\|u_0\|_{L^2}<1\}}}{\lambda^{\theta+1/2}\log(\frac{1}{\lambda^{1/2}\|u_0\|_{L^2}})}.\end{aligned} \tag{4.71}$$

By the proof of Lemma 2.1 the function $G(\theta)$ is nondecreasing, thus we see that

$$G(\delta_\lambda) \leqslant \left|G(1)\right| < 2\varsigma = -\log\lambda^2,$$

consequently

$$\left|G(\delta_\lambda)+\log\lambda^2\right| = 2\varsigma - G(\delta_\lambda) \geqslant 2\varsigma - \left|G(1)\right| = 1$$

and

$$\lambda^{k_\alpha\delta_\lambda} = e^{-\varsigma k_\alpha\delta_\lambda} \leqslant \exp\left\{-\frac{k_\alpha\delta_\lambda}{2}\frac{\Lambda_{\delta_\lambda}(u_0)}{\|u_0\|_{\dot H^{\delta_\lambda}}^2}\right\},$$

and also we have

$$\log\left(\frac{1}{\lambda^{1/2}\|u_0\|_{L^2}}\right) = \frac{1}{2}\log\left(\frac{1}{\lambda\|u_0\|_{L^2}^2}\right) = \frac{1}{4}\log\left(e\frac{e^{|G(1)|}}{\|u_0\|_{L^2}^4}\right),$$

therefore by (4.71) for all $\theta \in [0,1]$ we obtain

$$\begin{aligned}\left\|u(t)\right\|_{\dot H^\theta} &\leqslant \kappa_\alpha\|u_0\|_{\dot H^\theta} + c_\alpha\lambda^{1-\theta}\|u_0\|_{L^2}^3 + \lambda^{1-\theta}\|u_0\|_{L^2}^3\\ &\quad + c_\alpha\|u_0\|_{L^2}^\alpha\frac{e^{|G(1)|(\theta/2+1/4-\alpha/4)-\alpha/4}}{\log(e\frac{e^{|G(1)|}}{\|u_0\|_{L^2}^4})}\chi_{\{\lambda^{1/2}\|u_0\|_{L^2}<1\}}.\end{aligned}$$

We conclude the proof of the lemma. □

Lemma 4.11. *Let $u_0 \in H^{k+1}$ and $u(t)$ be the solution of some differential equation satisfying* (1.1)–(1.3)*, then we have*

$$\|u(t)\|_{\dot H^\theta} \leqslant n\|u_0\|_{\dot H^\theta} + c\|u_0\|_{L^2}^{1+(n-1)\theta/k} + c\|u_0\|_{L^2}^{1+(n-1)\theta/k} e^{\{(2|G(k)|-G(k)+|\log(\|u_0\|_{L^2}^{2(n-1)/k})|)(\theta/2+k/\{2(n-1)\})\}}. \tag{4.72}$$

Proof. Inequality (4.72) is a consequence of inequality (4.67) by scaling. In fact, we use Lemma 4.10 ($\alpha = 0$) for the scaling solution (1.2), with

$$\lambda = \frac{\exp\{\frac{-(|G(k)|+|\log\|u_0\|_{L^2}^{(n-1)/k}|)}{2\beta_2}\}}{\|u_0\|_{L^2}^{(n-1)/(2\beta_2 k)}},$$

then λ is such that

$$\log\lambda^{2\beta_2} = -|G(k)| - |\log\|u_0\|_{L^2}^{(n-1)/k}| - \log\|u_0\|_{L^2}^{(n-1)/k}, \tag{4.73}$$

and $\tilde G(k) = G(k) + \log\lambda^{2\beta_2} \leqslant 0$, thus

$$\begin{aligned}
\|u(t)\|_{\dot H^\theta} &\leqslant n\|u_0\|_{\dot H^\theta} + ce^{-|G(k)+\log\lambda^{2\beta_2}|(k-\theta)/2}\lambda^{nr(0)-r(\theta)}\|u_0\|_{L^2}^n + c\frac{e^{|G(k)+\log\lambda^{2\beta_2}|(\theta/2+k/\{2(n-1)\})}}{\lambda^{r(\theta)}\log(\frac{e^{|G(k)+\log\lambda^{2\beta_2}|+1}}{\lambda^{2r(0)(n-1)/k}\|u_0\|_{L^2}^{2(n-1)/k}})} \\
&= n\|u_0\|_{\dot H^\theta} + ce^{G(k)(k-\theta)/2}\lambda^{2\beta_2(k-\theta)}\|u_0\|_{L^2}^n + c\frac{(\frac{e^{-G(k)}}{\lambda^{4\beta_2}})^{\theta/2+k/\{2(n-1)\}}}{\log(\frac{e^{-G(k)}}{\lambda^{4\beta_2}}\frac{e}{\|u_0\|_{L^2}^{2(n-1)/k}})} \\
&\leqslant n\|u_0\|_{\dot H^\theta} + ce^{-|\log\|u_0\|_{L^2}^{(n-1)/k}|(k-\theta)}\|u_0\|_{L^2}^{1+(n-1)\theta/k} \\
&\quad + c\|u_0\|_{L^2}^{1+(n-1)\theta/k}\frac{e^{\{(2|G(k)|-G(k)+|\log(\|u_0\|_{L^2}^{2(n-1)/k})|)(\theta/2+k/\{2(n-1)\})\}}}{1+2|G(k)|-G(k)+|\log(\|u_0\|_{L^2}^{2(n-1)/k})|},
\end{aligned} \tag{4.74}$$

where in the second inequality we used that

$$nr(0) - r(\theta) = \beta_2(k-\theta) \quad \text{and} \quad r(\theta) = \beta_2\theta + \beta_2 k/(n-1),$$

therefore (4.74) gives (4.72). □

Lemma 4.12. *Let $u_0 \in H^{k+1}$ and $u(t)$ be the solution of some differential equation satisfying* (1.1)–(1.3)*, then we have*

$$\|u(t)\|_{\dot H^\theta} \leqslant n\|u_0\|_{\dot H^\theta} + c\|u_0\|_{L^2}^{1+(n-1)\theta/k} + c\|u_0\|_{L^2}^{1+(n-1)\theta/k} e^{\{|G(k)-\log(\|u_0\|_{L^2}^{2(n-1)/k})|(\theta/2+k/\{2(n-1)\})\}}. \tag{4.75}$$

Proof. The proof is similar with the proof of Lemma 4.11, using scaling with

$$\lambda = e^{-G(k)/(2\beta_2)}. \qquad \square$$

Now, the proof of Theorem 1.1 is a consequence of Lemmas 4.7 and 4.12.

4.2. Proof of Theorem 1.2

Let $u(t)$ a solution of the model (1.11)–(1.13). Without lose of generality we suppose that $u(t)$ is a solution of complex mKdV equation. We need the following result:

Lemma 4.13. *Let $u_0 \in H^2(\mathbb{R})$, $\|u_0\|_{L^2}^4 \geqslant e^{G(1)}$ and $u(t)$ be the solution of the complex m-KdV* (1.4) *with initial data u_0, then*

$$\|u(t)\|_{\dot H^\theta} \leqslant \|u_0\|_{\dot H^\theta} + c\max\{\tilde k_0^\theta, 1\}\|u_0\|_{L^2}^{1+2\theta}, \tag{4.76}$$

where $\tilde k_0 = -(e+d)/(6b) > 0$.

Proof. Let

$$v(x,t) = \tilde k_0^{1/2}u(x,t),$$

then

$$G_{v_0}(1) = G_{u_0}(1), \quad \|v_0\|_{\dot H^\theta} = \tilde k_0^{1/2}\|u_0\|_{\dot H^\theta}$$

and v satisfies

$$\partial_t v + b\partial_x^3 v + \frac{d}{\tilde{k}_0}|v|^2\partial_x v + \frac{e}{\tilde{k}_0}v^2\partial_x\bar{v} = 0,$$

inequality $\|u_0\|_{L^2}^4 \geqslant e^{G(1)}$ is equivalent with $\|v_0\|_{L^2}^4/\tilde{k}_0^2 \geqslant e^{G_{v_0}(1)}$.

We consider two cases:

(1) If $\tilde{k}_0 \geqslant 1$. In this case we have $\|v_0\|_{L^2}^4 \geqslant e^{G_{v_0}(1)}$ and Lemma 4.7 (with $\mu = 1$ and $|k_0| = 1$) gives

$$\|v(\tilde{k}_0 t)\|_{\dot{H}^\theta} \leqslant \|v_0\|_{\dot{H}^\theta} + c\|v_0\|_{L^2}^{1+2\theta},$$

and from this inequality follows (4.76).

(2) If $\tilde{k}_0 < 1$. Let $\mu = 1/\tilde{k}_0 > 1$, in this case $\|v_0\|_{L^2}^4\mu^2 \geqslant e^{G_{v_0}(1)}$ and Lemma 4.7 (with $\mu = 1/\tilde{k}_0 > 1$ and $|k_0| = 1$) gives

$$\|v(\tilde{k}_0 t)\|_{\dot{H}^\theta} \leqslant \|v_0\|_{\dot{H}^\theta} + c\frac{1}{|\tilde{k}_0|^\theta}\|v_0\|_{L^2}^{1+2\theta},$$

and this inequality also implies (4.76). $\square$

Now by Lemma 4.13, it is sufficient to consider the case when

$$\|u_0\|_{L^2}^4 < e^{G(1)}. \tag{4.77}$$

Let

$$\lambda = \frac{\|u_0\|_{L^2}^2}{e^{G(1)/2}}, \tag{4.78}$$

and let $v(x,t) = (1/\lambda)u(x,t)$, then $G_{v_0}(1) = G_{u_0}(1) = G(1)$ and v satisfies

$$\partial_t v + b\partial_x^3 v + \lambda^2 d|v|^2\partial_x v + \lambda^2 e v^2\partial_x\bar{v} = 0,$$

inequality (4.78) implies that

$$\lambda^4 = \frac{e^{2G(1)}}{\|v_0\|_{L^2}^8}. \tag{4.79}$$

Combining (4.77) and (4.79) we have

$$e^{G(1)} = e^{G_{v_0}(1)} < \|v_0\|_{L^2}^4,$$

therefore Lemma 4.13 gives

$$\|v(t)\|_{\dot{H}^\theta} \leqslant \|v_0\|_{\dot{H}^\theta} + c\max\{\lambda^{2\theta}|k_0|^\theta, 1\}\|v_0\|_{L^2}^{1+2\theta},$$

from this inequality and definition of v, we obtain (1.15).

Appendix A

Proof of inequality (4.58). If $\theta = 0$ we use the quantity conserved in L^2 and if $\theta = 1$ the result is a consequence of Corollary 3.3. In fact, by (3.34) we have

$$\|v(t)\|_{H^1}^2 \leqslant 2\|v(0)\|_{H^1}^2 + 2\|v(0)\|_{L^2}^2 + c\|v(0)\|_{L^2}^6,$$

and this inequality implies

$$\|v(t)\|_{H^1}^2 - \|v(0)\|_{H^1}^2 \leqslant (1+N^2)\|v(0)\|_{L^2}^2 + 2\|v(0)\|_{L^2}^2 + c\|v(0)\|_{L^2}^6.$$

Now, for t fixed, we define the function

$$f_t(\delta) := \|v(t)\|_{H^\delta}^2 - \|v(0)\|_{H^\delta}^2, \quad \delta \in [0,1],$$

and we consider

$$f_t(\delta_0) = \max_{\delta\in[0,1]} f_t(\delta).$$

If $\delta_0 = 0$ or $\delta_0 = 1$, we have (4.58). If $\delta_0 \in (0,1)$ then $f_t'(\delta_0) = 0$, therefore

$$\int_{\mathbb{R}} (1+\xi^2)^{\delta_0}\log(1+\xi^2)|\widehat{v}(t,\xi)|^2\,d\xi = \int_{\mathbb{R}} (1+\xi^2)^{\delta_0}\log(1+\xi^2)|\widehat{v}(0,\xi)|^2\,d\xi. \tag{A.80}$$

From (2.27) and (A.80) we have

$$\begin{aligned}\|v(t)\|_{H^{\delta_0}}^2 &\leqslant \|v(0)\|_{L^2}^2 \exp\left\{\frac{\delta_0\int(1+\xi^2)^{\delta_0}\log(1+\xi^2)|\widehat{v}(t,\xi)|^2\,d\xi}{\|v(t)\|_{H^{\delta_0}}^2}\right\}\\ &= \|v(0)\|_{L^2}^2 \exp\left\{\frac{\delta_0\int(1+\xi^2)^{\delta_0}\log(1+\xi^2)|\widehat{v}(0,\xi)|^2\,d\xi}{\|v(t)\|_{H^{\delta_0}}^2}\right\}\\ &\leqslant \|v(0)\|_{L^2}^2 \exp\left\{\frac{\delta_0\log(1+N^2)\int(1+\xi^2)^{\delta_0}|\widehat{v}(0,\xi)|^2\,d\xi}{\|v(t)\|_{H^{\delta_0}}^2}\right\}\\ &= \|v(0)\|_{L^2}^2\big(1+N^2\big)^{\frac{\delta_0\|v(0)\|_{H^{\delta_0}}^2}{\|v(t)\|_{H^{\delta_0}}^2}}.\end{aligned}$$

Let $\chi(t)=\|v(t)\|_{H^{\delta_0}}^2$, $L_0=\|v(0)\|_{L^2}^2$, $q=\langle N^2\rangle^{\delta_0}$ and $k_0=q\delta_0/(e\log q)$, then

$$\frac{\chi(t)}{L_0}\leqslant q^{\frac{\chi(0)}{\chi(t)}},$$

taking the logarithm in the above inequality we have

$$\frac{\chi(t)}{L_0}\log\frac{\chi(t)}{L_0}\leqslant\frac{\chi(0)}{L_0}\log q, \tag{A.81}$$

and by (4.41) and (A.81), this implies

$$\frac{\chi(t)}{L_0}\log\frac{\chi(t)}{L_0}\leqslant\left(\frac{\chi(0)}{L_0}+k_0+qe\right)\log\left(\frac{\chi(0)}{L_0}+k_0+qe\right). \tag{A.82}$$

Since the function $g(x)=x\log x$, $x>1$ is nondecreasing, by the definition of δ_0 we obtain (4.58). □

References

[1] T.B. Benjamin, Internal waves of permanent form in fluids of great depth, J. Fluid Mech. 29 (1967) 559–592.
[2] A. Hasegawa, Y. Kodama, Nonlinear pulse propagation in a monomode dielectric guide, IEEE J. Quantum Electron. 23 (1987) 510–524.
[3] Y. Kodama, Optical solitons in a monomode fiber, J. Stat. Phys. 39 (1985) 597–614.
[4] H. Ono, Algebraic solitary waves in stratified fluids, J. Phys. Soc. Japan 39 (1975) 1082–1091.